QN.- State and Raabe's Test for the Converges of a Series of positive terms.

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Statement of Raabe's Test:-

If $\sum a_n$ is a Series of positive terms such that,

$$\lim_{n\to\infty}\left\{n\left(\frac{a_n}{a_{n+1}}-1\right)\right\} \to L.$$

then, (i) $\sum a_n$ is Convergent if $L>1$.
(ii) $\sum a_n$ is divergent if $L<1$.
(iii) and the test fails when $L=1$.

Proof:- To Prove this theorem Let us Compare the given series $\sum a_n$ with an auxiliary Series $\sum b_n$.

$$\text{where } \sum b_n = \sum \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \text{-----} \infty.$$

As we know that $\sum b_n$ is
(1) Convergent when $p>1$
and divergent when $p\leq 1$.

Hence, from Comparison Test the given Series $\sum a_n$ is Convergent or divergent according as,

$$\frac{a_n}{a_{n+1}} > \text{ or } < \frac{b_n}{b_{n+1}}.$$

$$\text{or, } \frac{a_n}{a_{n+1}} > \text{ or } < \frac{\frac{1}{n^p}}{\frac{1}{(n+1)^p}}$$

$$\text{or, } \frac{a_n}{a_{n+1}} > \text{ or } < \frac{(n+1)^p}{n^p}$$

$$\text{or. } \frac{a_n}{a_{n+1}} > \text{ or } < \left(\frac{n+1}{n}\right)^p$$

$$\text{or } \frac{a_n}{a_{n+1}} > \text{ or } < \left(\frac{n\left(1+\frac{1}{n}\right)}{n}\right)^p$$

$$\text{or } \frac{a_n}{a_{n+1}} > \text{ or } < \left(1+\frac{1}{n}\right)^p.$$

$$\text{or } \frac{a_n}{a_{n+1}} > \text{ or } < 1 + p\cdot\frac{1}{n} + \frac{p\cdot(p-1)}{1\cdot 2}\cdot\left(\frac{1}{n}\right)^2 + \cdots$$

$$\text{or } \frac{a_n}{a_{n+1}} - 1 > \text{ or } < \frac{1}{n}\left[p + \frac{p(p-1)}{1\cdot 2}\cdot\frac{1}{n} + \cdots\right]$$

$$\text{or } n\left(\frac{a_n}{a_{n+1}} - 1\right) > \text{ or } < p + \frac{p(p-1)}{1\cdot 2}\cdot\frac{1}{n} + \cdots$$

on taking limit when $n \to \infty$

$$\text{or } \lim_{n\to\infty}\left\{n\left(\frac{a_n}{a_{n+1}} - 1\right)\right\} > \text{ or } < p.$$

$$\text{or } \lim_{n\to\infty}\left\{n\left(\frac{a_n}{a_{n+1}} - 1\right)\right\} > \text{ or } < 1.$$

i.e. the given series is convergent.

when $\lim_{n\to\infty}\left\{n\left(\frac{a_n}{a_{n+1}} - 1\right)\right\} > 1.$

and divergent when $\lim_{n\to\infty}\left\{n\left(\frac{a_n}{a_{n+1}} - 1\right)\right\} < 1.$

and the test fails, when,

$$\underset{n\to\infty}{Lt}\left\{n\left(\frac{a_n}{a_{n+1}}-1\right)\right\}=1.$$

② QN: State and Prove Cauchy's Condensation Test.

Statement:- If $f(1), f(2) ---- f(n) --$ be a sequence of +ve terms such that

$f(1) \geq f(2) \geq f(3) \geq ---- \geq f(n) \geq -----$

for all n, Then the series $\sum f(n)$ & $\sum a^n . f(a^n)$ converges or diverges together (a is a +ve integer >1).

Proof:- As $\sum f n$ is a series of positive terms. Hence grouping of terms is allowed.

Now, grouping the series $\sum f(n)$ as follows.

$\{f(1) + f(2) + --- + f(a)\}$
$+\{f(a+1) + f(a+2) + ---- + f(a^2)\}$
$+\{f(a^2+1) + f(a^2+2) + --- + f(a^3)\}$
$+ ---------$
$+ ---------$
$+$

v_n = nth group of the series
$= f(a^{n-1}+1) + f(a^{n-2}+2) + ----- + f(a^n)\}$

∴ the no. of terms in the nth group is $a^n - a^{n-1}$ and since $f(n)$ is a decreasing function.

$f(a^n) \leq$ each term of V_n

and $f(a^{n-1}) \geq$ each term of V_n

$(a^n - a^{n-1}) f(a^n) \leq V_n \leq (a^n - a^{n-1}) f(a^{n-1})$

∴ $V_n \leq a^{n-1}(a-1) f(a^{n-1})$ — ①

$V_n \geq \frac{1}{a}(a-1)a^n \cdot f(a^n)$ —— (2)

Now, if $\sum a^n f(a^n)$ is Convergent, the $\sum a^{n-1} f(a^{n-1})$ is also Convergent by (1) and the Comparision Test, $\sum V_n$ is Convergent

$\therefore \sum f(n)$ is Convergent.

When $\sum a^n f(a^n)$ is divergent then from (2), and the Comparision Test $\sum V_n$ is divergent $\therefore \sum f(n)$ is divergent

Hence, a series of +ve terms is either Convergent or divergent.

$\therefore \sum f(n)$ and $\sum a^n \cdot f(a^n)$ is both Convergent or both divergent.